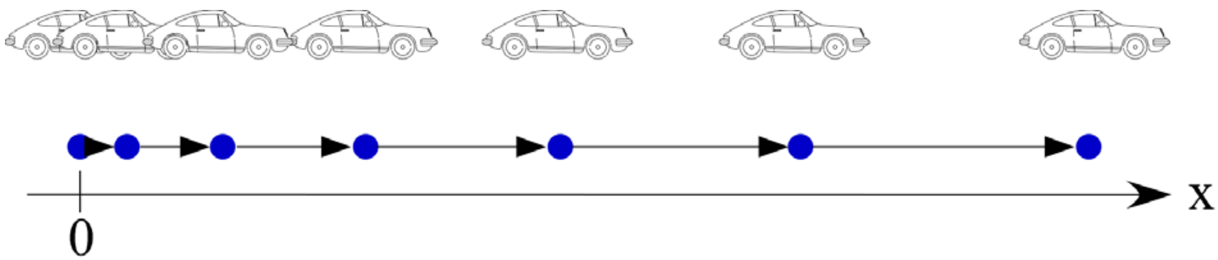
Reading Page: Motion Diagrams I

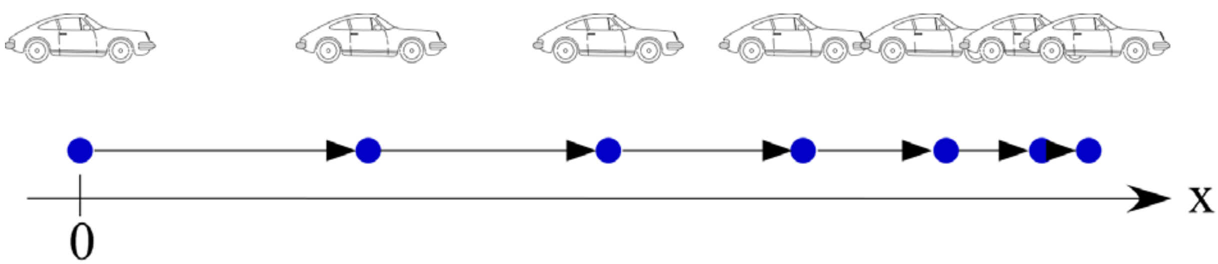
In the previous unit we learned how to draw a motion diagram for an object moving with a constant velocity by indicating the position and velocity of the object at equally spaced time intervals. When an object moves with increasing or decreasing speed, the distance between successive dots increases or decreases, respectively.

Examples of motion diagrams:

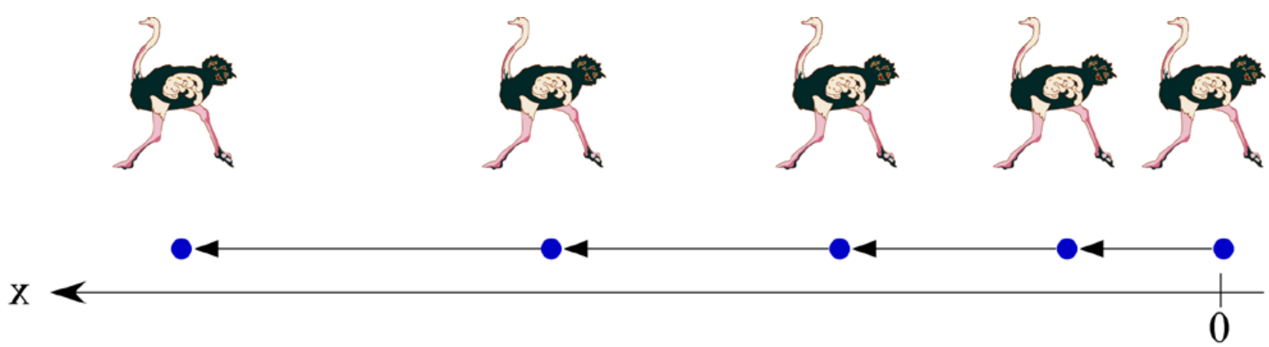
Example 1. A car starts from rest and speeds up. The position dots, which are drawn at regular clock ticks, get farther and farther apart. The distance between the dots increases, and the length of the velocity arrow also increases: the velocity of the car is increasing.



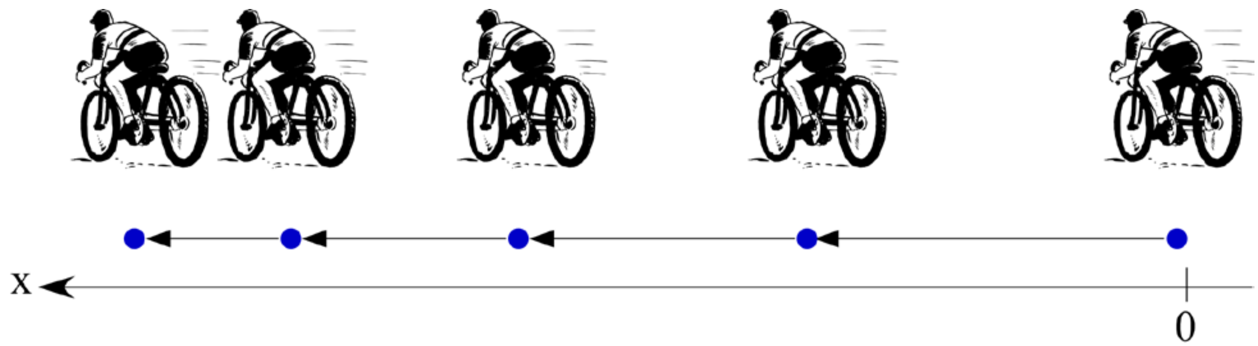
Example 2. A car slows down to a stop. As the distance between the dots decreases, the length of the velocity arrow also decreases: the velocity of the car is decreasing.



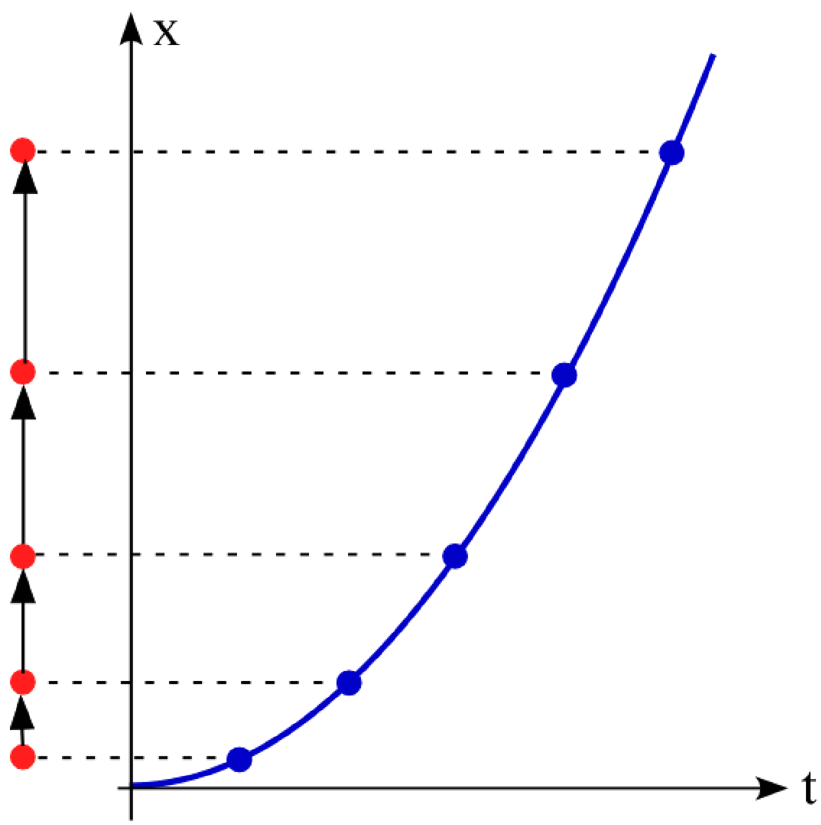
Example 3: An ostrich speeds up, trying to catch up with his group. The speed of the ostrich is increasing. Its position dots get farther apart as its speed increases. Notice that the x=0 position is on the right, and the ostrich moves toward the left.



Example 4: After participating in a 20 mile bike race, Tom is approaching his home. His speed decreases and he prepares to stop. x=0 is on the right, and he moves toward the left.



Example 5: Correlating graphs to motion diagrams.



Reading Page: Instantaneous Velocity: Geometric Method

Teacher notes: This Reading Page should be part of a whole-class discussion.

For uniform motion (i.e., at a constant speed) the x-t graph is a straight line. We can speak of the average velocity as the slope of the graph because the slope of a straight line is the same constant value everywhere. It doesn't matter whether we find the slope during a time interval at the beginning of the object's motion or the end: Slope 1 = Slope 2 (figs 1 and 2). Furthermore, the slope is the same at every instant of time during the object's uniform motion.

Fig 1.

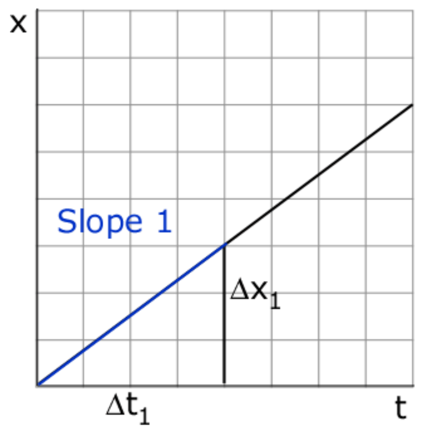
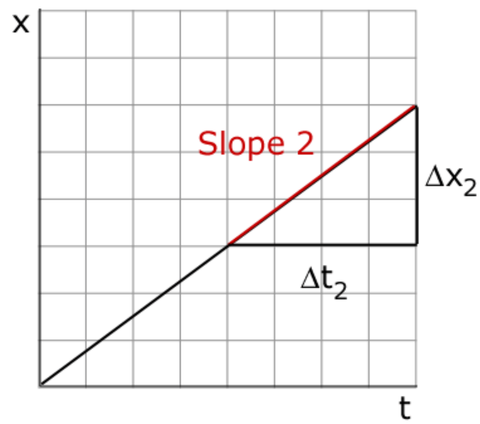


Fig. 2



When an object starts slowly and steadily increases its speed, the *x-t* graph is not a straight line. The slope over a time interval at the beginning of the journey is different from that at the end of the journey: Slope 1 calculated at an earlier time is not the same as Slope 2 at a later time (figs 3 and 4), even if the time interval is the same in both cases

Therefore, we need to decide the *clock reading* at which we want to calculate the slope.

Fig 3.

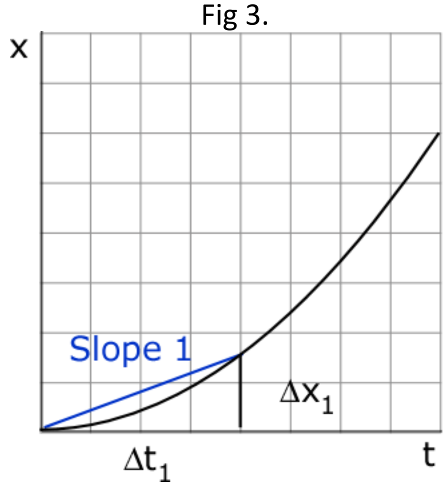
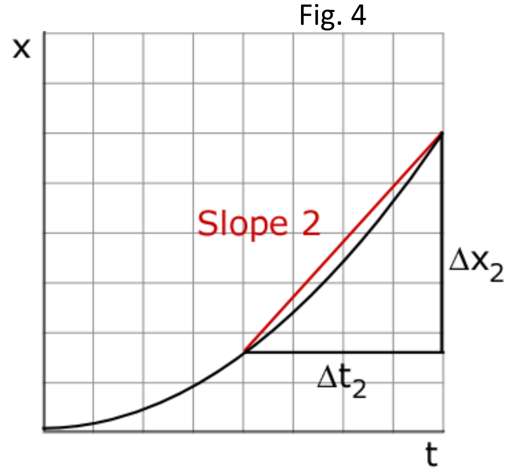


Fig. 4



If we want to calculate the slope at a clock reading t_1 we take a time interval Δt around the time t_1 (t_1 should be in the center of interval Δt). The slope of the secant is an approximation of the average velocity at t_1. (fig. 5)

As we make the time interval Δt smaller and smaller (figs 6 and 7), Δt finally shrinks to a tiny value or "an instant" (fig 8). When Δt becomes "an instant," the line of the slope just touches the curve at clock reading t_1, and the secant becomes the tangent to the curve at t_1.

The velocity given by the slope of the tangent is the instantaneous velocity AT time t_1, not just AROUND time t_1 (as the slopes of the secants gave us).

Fig. 5.

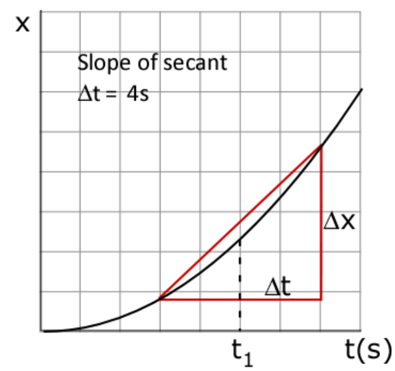


Fig. 6

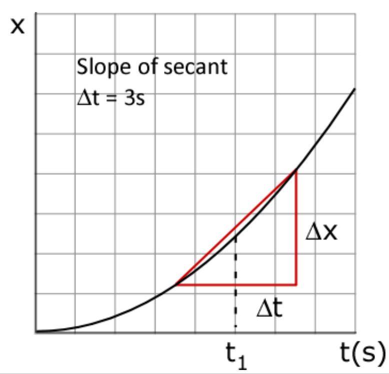


Fig. 7

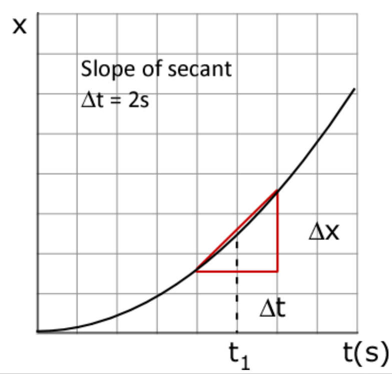
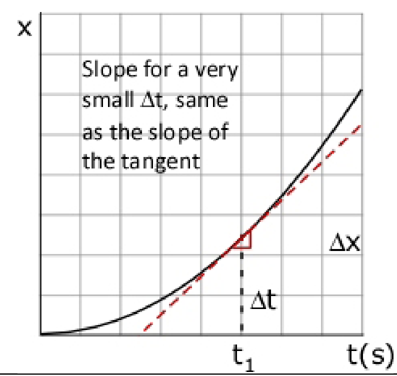


Fig. 8



For a smooth x-t curve, the slope of the tangent is equal to the slope of the secant for the same center point (fig. 9).

Why then, should we bother with tangents or instantaneous velocities when the average velocities give us the same values?

Well, using secants works for smooth curves. If we have a curve with kinks or bumps we need to use tangents to find the instantaneous velocity; we do not use secants, because they do not give the same slopes as tangents (fig. 10).

So long as we deal with smooth curves, we can use secants more conveniently.

Fig. 9

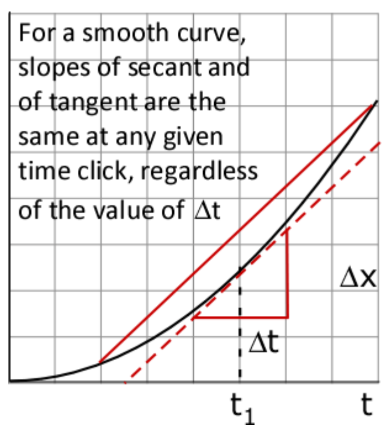
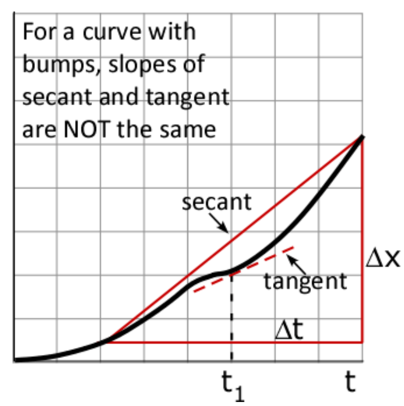


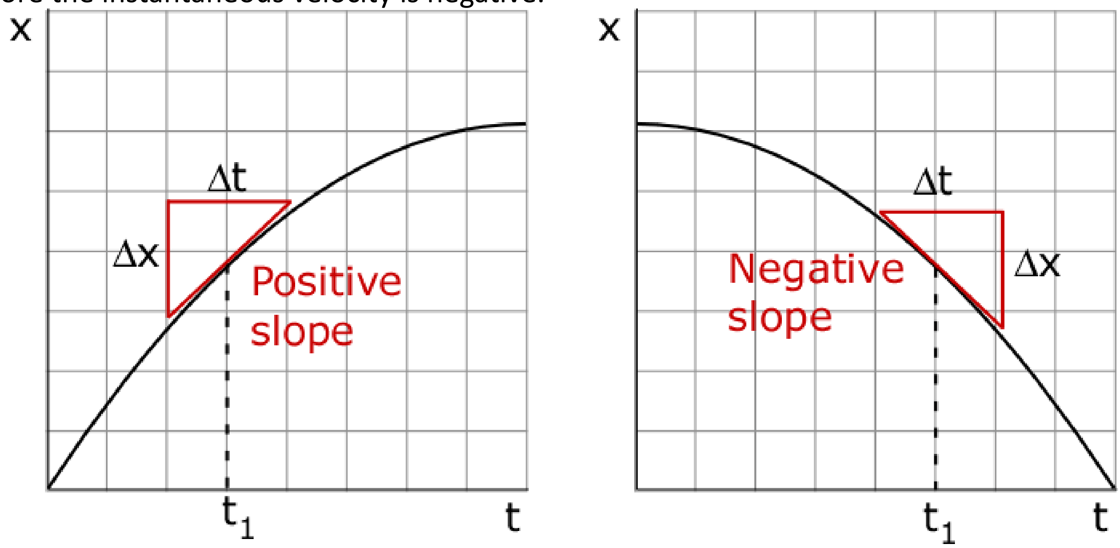
Fig. 10



Slopes can be positive or negative.

If x is increasing with time at the clock reading where we calculate the slope, the slope, and therefore the instantaneous velocity is positive.

If x is decreasing with time at the clock reading where we calculate the slope, the slope, and therefore the instantaneous velocity is negative.



Note: a secant is similar to a chord in a circle. In a circle, a straight line that connects any two points of the circumference is called a chord. For a general curve, a straight line that connects any two points on the curve is called a secant.

Powerpoint : Slopes.ppt

Reading Page: Numerical analysis of *x-t* data to obtain *v-t* data

We have learned how to figure out instantaneous velocities from the slopes when we have an x-t graph for accelerated motion. What if we have only a table of values?

x-t data for Jack's toy car	
t (s)	x (cm)
0.0	6.0
2.0	6.4
4.0	7.6
6.0	9.6
8.0	12.4
10.0	16.0
12.0	20.4
14.0	25.6
16.0	31.6
18.0	38.4

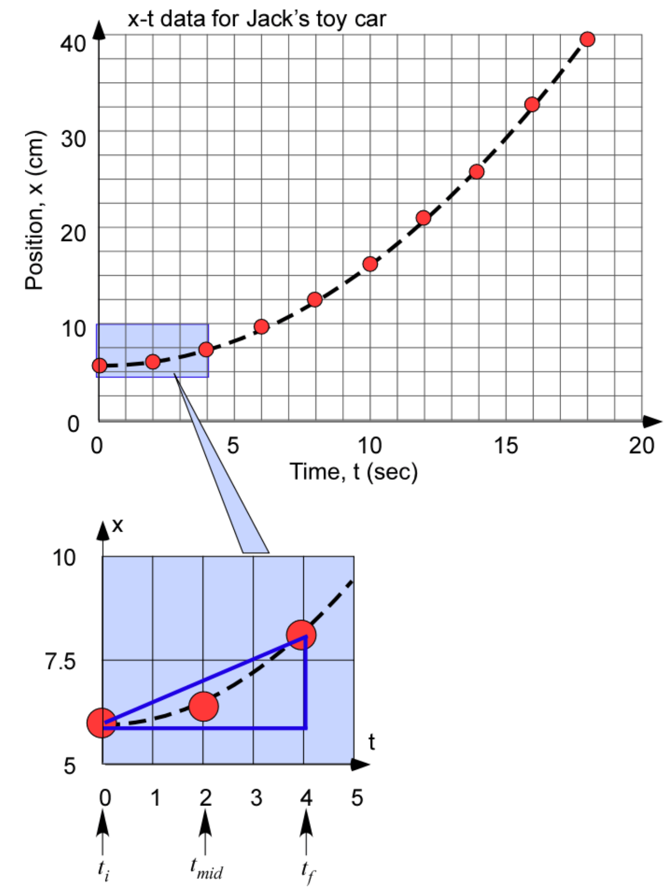
Example 1. Jack runs his toy car along the floor and obtains the data shown in the table.

a) Draw an x-t graph

b) Calculate the velocity of the car from the values in the table at t = 2 sec.

Solution:

(a)



(b)

From the figure we see that the slope is calculated using the usual rise/run formula. To obtain the slope at t = 2 sec, i.e., t_{mid} = 2 sec, we take an interval around that time, for example, from t_i = 0 to t_f = 4 sec; reading off the graph, x_f = 7.6 and x_i=6.0 cm.

Therefore the slope (which is the average velocity at t_{mid}=2 sec is

$$v_{inst} = \frac{x_f - x_i}{t_f - t_i}$$

$$v_{2\,\sec} = \frac{7.6-6}{4-0} = 0.4 cm/\sec$$

If we look back at the table, we see that the same values of position x_f at time t_f and position x_i at time t_i are also listed in the table. We can, in fact, just as well use the x and t values from the table, to find the velocities.

t (s)	x (cm)	v_{avg} in cm/s	At t_{mid} (s)
0.0	6.0	=(7.6-6.0) / (4-0)=0.4	=(4+0)/2 =2.0
2.0	6.4		
4.0	7.6		

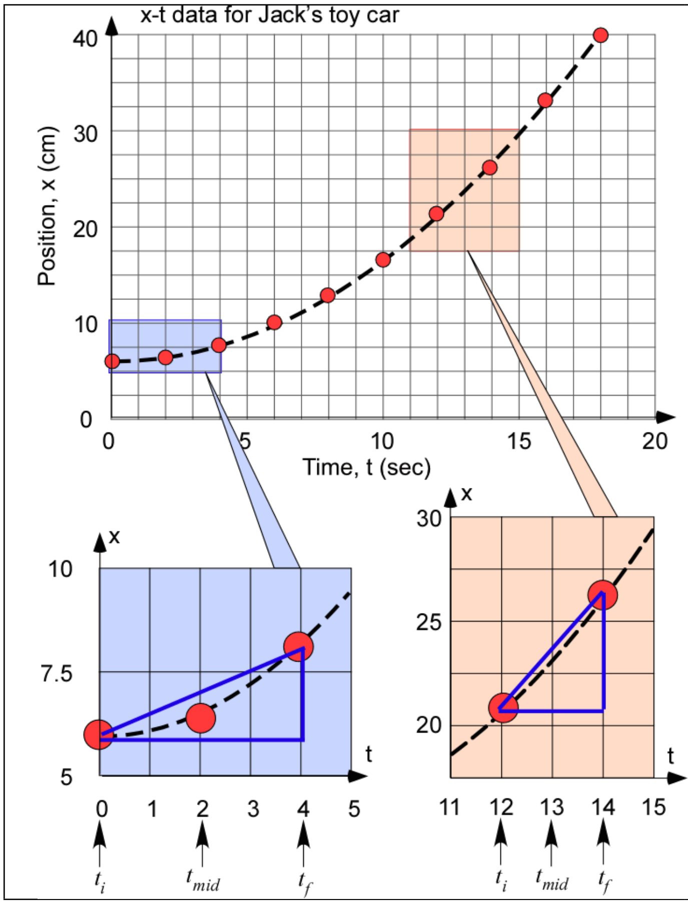
And this method is exactly equivalent to calculating the slopes using secants.

Notice that we called the velocity above the average velocity rather than the instantaneous velocity – because we used the secant and not the tangent. If the time interval Δt becomes smaller and smaller, the average velocity will approach the value of the instantaneous velocity in all cases. In the case above, since we have a smooth curve, the average and instantaneous velocity are the same even when the time interval is not infinitesimally small.

Example 2. Use the same data as in Example 1, and calculate the velocity at t = 13 sec.

Solution:

This method can also be used to calculate average velocity at time clicks between values for which we have data in the table. Below we'll calculate the velocity at t = 13 sec (a value not in the table) using the values around that time click.



The figure shows an expanded segment near t=13 sec. We can calculate the average velocity using the values at t_i = 12 sec and t_f = 14 sec using the table of x-t data. A small section of the table is shown below. Notice that we inserted a row between the data – this is just for convenience while we make calculations

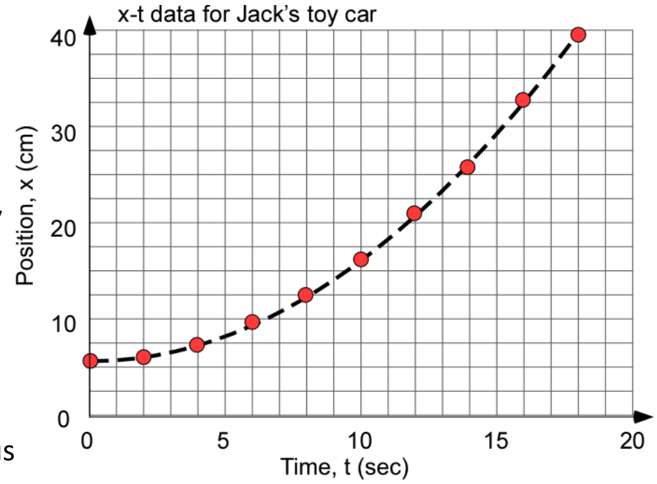
The velocity and midpoint time are calculated in the same manner as before.

t (s)	x (cm)	vavg in cm/s	At tmid (s)
12.0	20.4	=(25.6-20.4) / (14-12) =2.6 cm/s	=(14+12) /2 = 13 sec
14.0	25.6		

Example 3. Use the same data as above

a) Calculate the average velocity at several clock readings

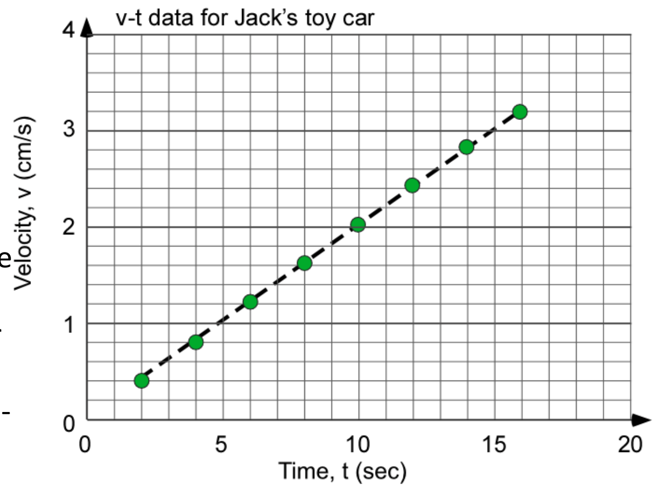
b) Using the midpoint time for the time interval chosen to calculate the average velocity and plot a v-t graph.



Solution:

(a) This value of the average velocity for the time interval from 0 s to 4 s was already shown. In a similar manner we can calculate the instantaneous velocity at t = 4 s, by choosing the time interval between t = 2 s and t = 6 s.

$$v_{t=4s} = \frac{x_6 - x_2}{t_6 - t_2} = \frac{9.6 - 6.4}{6 - 2} = 0.8 cm/s$$



This calculation is shown in the table below.

(b) The values for the instantaneous velocity in the table above were obtained as shown in the calculations. Note: Notice that you need two values for the time and accordingly two values for the position of the object to obtain an average or instantaneous velocity.

(c) Using the velocity and t_{mid} data from the table below, one can plot the v vs t graph as shown.

t (s)	x (cm)	v_{avg} in cm/s "instantaneous velocity"	Midpoint time t_{mid} (s)
0.0	6.0		
2.0	6.4	=(7.6-6.0)/(4-0)=0.4	(4+0)/2=2.0
4.0	7.6	= (9.6-6.4)/(6-2)=0.8	4.0
6.0	9.6	1.2	6.0
8.0	12.4	1.6	8.0
10.0	16.0	2.0	10.0
12.0	20.4	2.4	12.0
14.0	25.6	2.8	14.0
16.0	31.6	3.2	16.0
18.0	38.4		

Example 4: Kala runs a race at her school, and the following data is recorded for her motion.

(a) Calculate her instantaneous velocity along her path at several moments in time.

x vs. t data for Kala	
Time (s)	Distance (m)
0	0
3	12
6	26
9	41
12	58
15	78
18	100
21	120
24	136
27	150
30	162

In **Example 2** we determined the instantaneous velocity at a clock tick value by calculating the average velocity over a time interval that is centered on that clock tick value. For that, we used position data from the two neighboring clock-tick values. If we use x at t = 3s and x at t = 0s to calculate v_{inst}, we will obtain the value of v_{inst} at the clock reading half-way between t = 3s and t = 0s, namely at t = 1.5s. This is shown in the first row of the table below. Similarly, using x at t = 3s and x at t= 6s gives us v_{inst} at t = 4.5s.

The first two rows indicate how her velocity is calculated.

Time (s)	Position (m)	Instantaneous velocity (m/s)	At clock reading t_{mid} (s)
0	0		
		v=(12-0)/(3-0)=4 m/s	t=(3+0)/2=1.5 s
3	12		
		v=(26-12)/(6-3)=4.67 m/s	t=(6+3)/2=4.5 s
6	26		
		5.0	
9	41		
		5.67	
12	58		
		6.67	
15	78		
		7.33	
18	100		
		6.67	
21	120		
		5.33	
24	136		
		4.67	
27	150		
		4.0	
30	162		

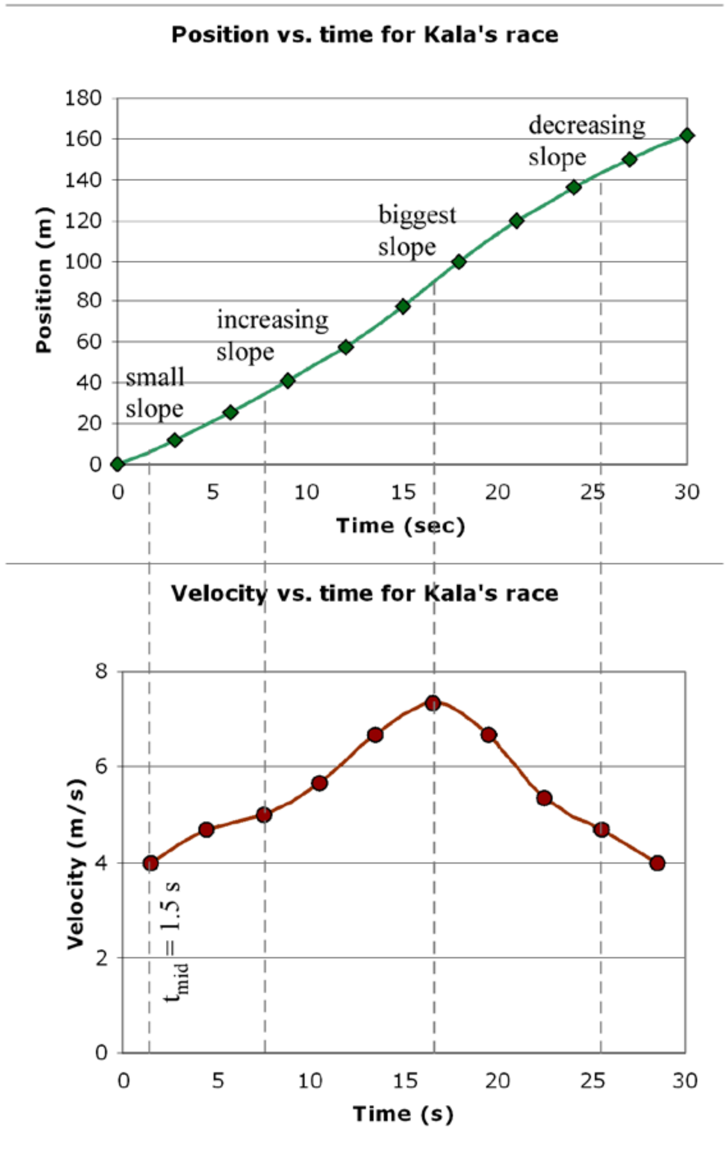
Notice that the table is arranged so that time and position are entered in alternate rows, and instantaneous velocity and midpoint clock reading are entered in the rows inbetween. This is done for convenience.

Shown next are the position-time graph and the velocity-time graph

Position-time graph for Kala's race. Notice that the entire curve is not a straight line, but the segments between successive points seem like straight lines. Also, each of the individual segments has a slightly different slope from the neighboring segments. These different slopes are due to the different speeds, and that is reflected in the v vs t graph shown below.

The varying slopes of individual segments in the x vs t graph are seen in the velocity-time graph. The first slope we can calculate is between t =0 and t = 3 sec; this slope gives us the velocity at the midpoint in time: t=1.5 sec.

The *slope* of the x-t graph between t_1 and t_2 gives the *value* of the v-t graph at the midpoint between t_1 and t_2, namely at $(t_1+t_2)/2$.

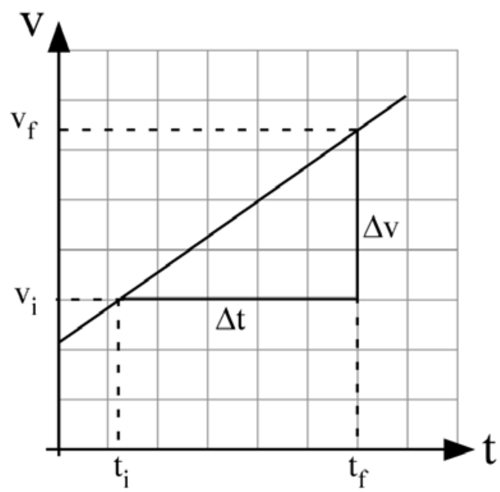


Reading Page: Acceleration

Teacher Notes: Use the first part of this Reading Page as a whole-class discussion.

Acceleration occurs when the velocity of an object changes with time. Acceleration (symbol: a) is given by the formula

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Time over which the change in velocity occurred}}$$



What units do we use for acceleration?

Since $a = \dfrac{v_f - v_i}{t_f - t_i}$ or $a = \dfrac{\Delta v}{\Delta t}$

Δv is in units of velocity or length/time; Δt is in units of time. Therefore acceleration has units of:

$$a = \frac{\Delta v}{\Delta t} = \frac{\text{length/time}}{\text{time}} = \frac{\text{length}}{(\text{time})(\text{time})} = \frac{\text{length}}{\text{time}^2}$$

Or

From the slope of the graph, we can write the acceleration to be

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

$$a\underbrace{(t_f - t_i)}_{\Delta t} = v_f - v_i$$

$$a\Delta t = v_f - v_i$$

$$a\Delta t + v_i = v_f$$

Which we can rearrange as

$$v_f = v_i + a\Delta t \text{ ... Motion Equation \# 1}$$

Here v_f, is the final velocity, v_i is the initial velocity, Δt is the time interval over which the velocity changes from v_i to v_f and a is the acceleration. The slope of the v vs t graph indicates the rate of change of velocity with time. If the slope is 2 m/s^2, that means that for every second the object is traveling its velocity is increasing by 2 m/s for every second.

Notice that Motion Equation #1 is similar to the equation for a straight line: $y = mx + b$

Comparing the two equations, we see that:

$$\left.\begin{array}{l} v_f = v_i + a\Delta t \\ y = mx + b \end{array}\right\} \Rightarrow \begin{cases} v_f = y & \text{final velocity} \\ v_i = b & \text{initial velocity (y intercept)} \\ a = m & \text{acceleration (slope)} \\ \Delta t = x & \text{time interval} \end{cases}$$

Since the v *vs* t graph is a straight line, its equation can be written in the form $y = mx + b$ or in the form $v_f = a\Delta t + v_i$.

Example 1: A car accelerates from an initial velocity of 3 m/s to a final velocity of 11 m/s in a time of 4 sec. Calculate its acceleration.

Since v_i = 3 m/s, v_f = 11 m/s and t = 4 s, the acceleration is

$$a = \frac{v_f - v_i}{\Delta t} = \frac{(11 - 3)\text{ m/s}}{4\text{ s}} = 2\ \frac{\text{m}}{\text{s}^2}$$

What does the value 2 m/s^2 mean? It might make more sense if we write it as 2 m/s/s (read as 2 meters per second per second). This means that for *every* second that goes by, the car's velocity increases by 2 m/s. In the first second of the car's motion, its velocity changes (increases) by +2 m/s, from a speed of 3 m/s to 5 m/s. In the second second, it increases again from 5 m/s to 7 m/s. In the third second, it increases yet again, from 7 m/s to 9 m/s. And in the fourth and final second, it increases from 9 m/s to 11 m/s. In other words, the change in the car's velocity is 2 m/s *per second*.

Example 2. A truck has an initial velocity of 22 m/s. Over a time interval of 3 seconds, its velocity decreases to 13 m/s. Calculate its acceleration.

Since v_i = 22 m/s, v_f = 13 m/s and t = 3 s, the acceleration is

$$a = \frac{v_f - v_i}{\Delta t} = \frac{(13 - 22)\text{ m/s}}{3\text{ s}} = -3\ \frac{\text{m}}{\text{s}^2}$$

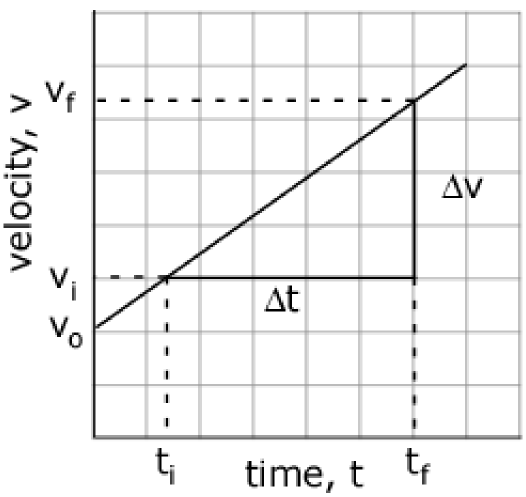
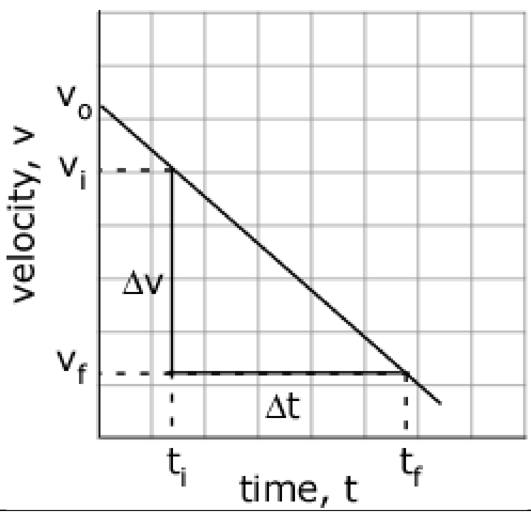
The truck's acceleration is –3 m/s^2. Notice that it has a negative sign. This negative sign indicates that the object has a negative value of acceleration. For every second that goes by the truck's velocity changes by –3 m/s. In other words, in the first second, the truck's velocity changes from 22 m/s to 22-3 = 19 m/s. In the second second, the velocity changes from 19 m/s to 19-3 = 16 m/s; finally, in the third second, from 16 m/s to 12-3 = 13 m/s.

The negative sign of the acceleration, when combined with the positive sign of the velocity, makes the velocity decrease with time.

Positive and negative accelerations

So far, most of the examples have had velocity increasing with time, similar to what happens when we step on the accelerator of a car. However, when we step on the brake, velocity decreases with time. While the word deceleration is often used in everyday language, we just use the general term "accelerated motion" when an object's velocity changes with time.

Here's what the graphs look like for object A, whose velocity increases with time, and object B, whose velocity decreases with time. Notice that both these objects have positive velocities.

Object A has a positive velocity:	Object B has a positive velocity:
	
Object A's velocity is positive. Its final velocity v_f is *more* than its initial velocity v_i; therefore the slope of the graph is positive and the acceleration *a* is positive. Since the velocity and the acceleration have the same signs (both are positive), object A *speeds up*.	Object B's velocity is positive. Its final velocity v_f is *less* than its initial velocity v_i; therefore the slope of the graph is negative and the acceleration *a* is negative. Since the velocity and the acceleration have opposite signs (*v* is positive while a is negative), object B *slows down*.

Slopes of Straight-line graphs

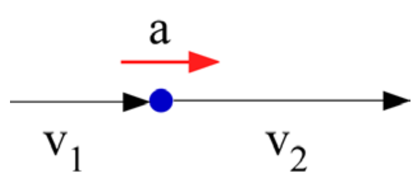
We have seen straight-line graphs before in uniform motion. When we have straight-line *x vs t* graphs, the position changes uniformly with time, and the *rate of change of position* gave us the *average velocity.*

In this unit, we see that the *v vs t* graphs are straight lines – namely, the velocity changes uniformly with time. In an analogous fashion, the *rate of change of velocity with time* is the *acceleration*.

Reading Page: Motion Diagrams II

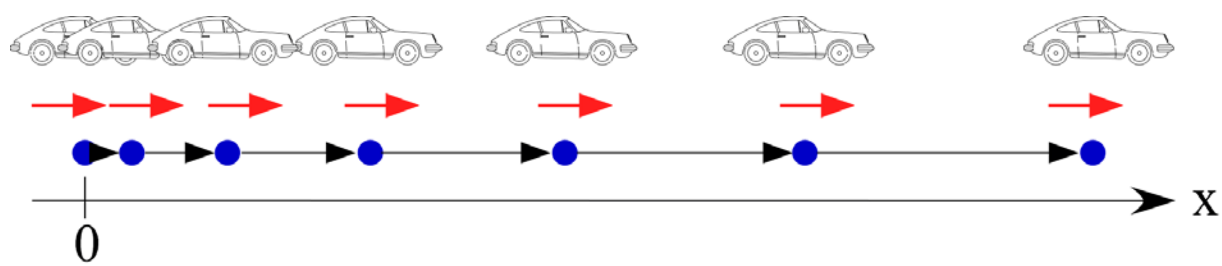
Motion diagrams with constant acceleration.

In addition to the position and velocity of the object we can also represent the acceleration of the object on the motion diagram. Since the acceleration is the change in velocity over a given time interval, and since we draw motion diagrams at regular clock ticks, the acceleration is proportional to the difference between the lengths of successive velocity arrows. Since the acceleration involves two velocities (initial and final), we draw the arrows between those two velocity vectors. The length of the acceleration vector is proportional to the difference between the two velocity vectors.



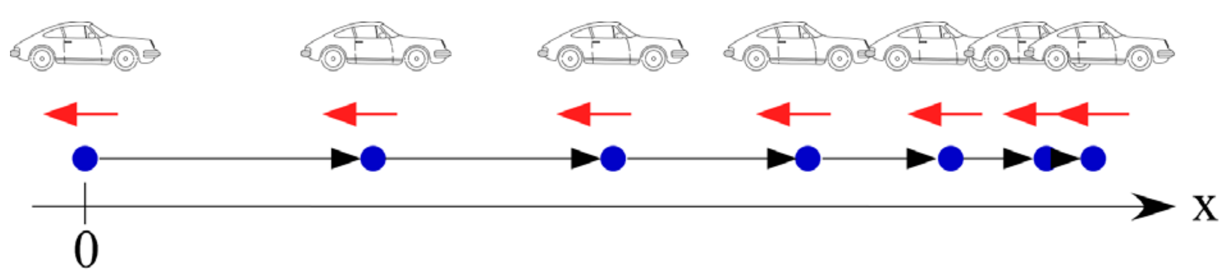
Examples of motion diagrams with constant acceleration:

Example 1. A car starts from rest and speeds up. The direction of the acceleration is in the same direction as the direction of its velocity.

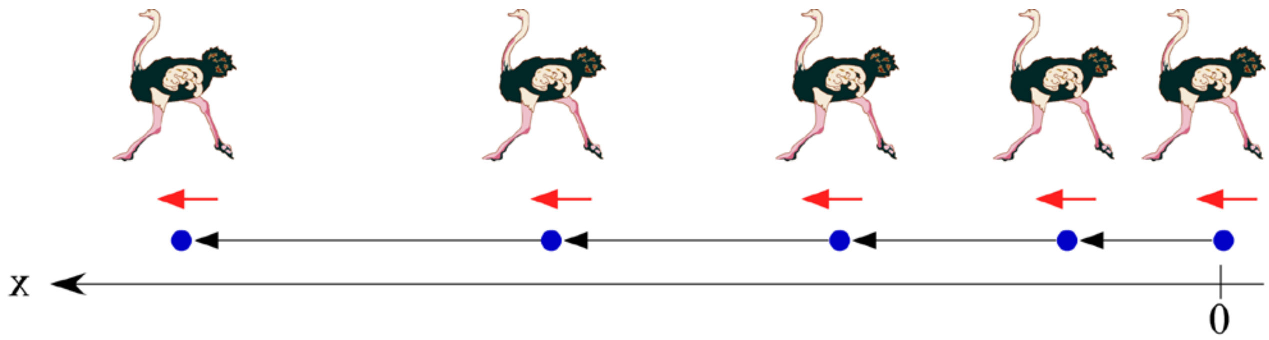


Note that we need to know two velocities to find the acceleration. Technically, we don't know the acceleration at the first and last points. However, since we will only deal with motion with constant acceleration, it is reasonable to expect the same acceleration at both first and last points, therefore it is ok to draw acceleration arrows above those points.

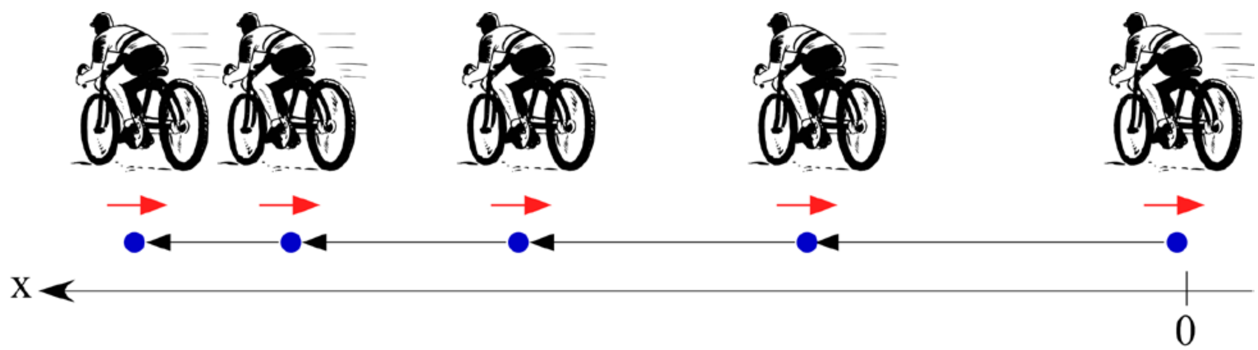
Example 2. A car slows down to a stop. The direction of the acceleration is opposite to the direction of the car's velocity.



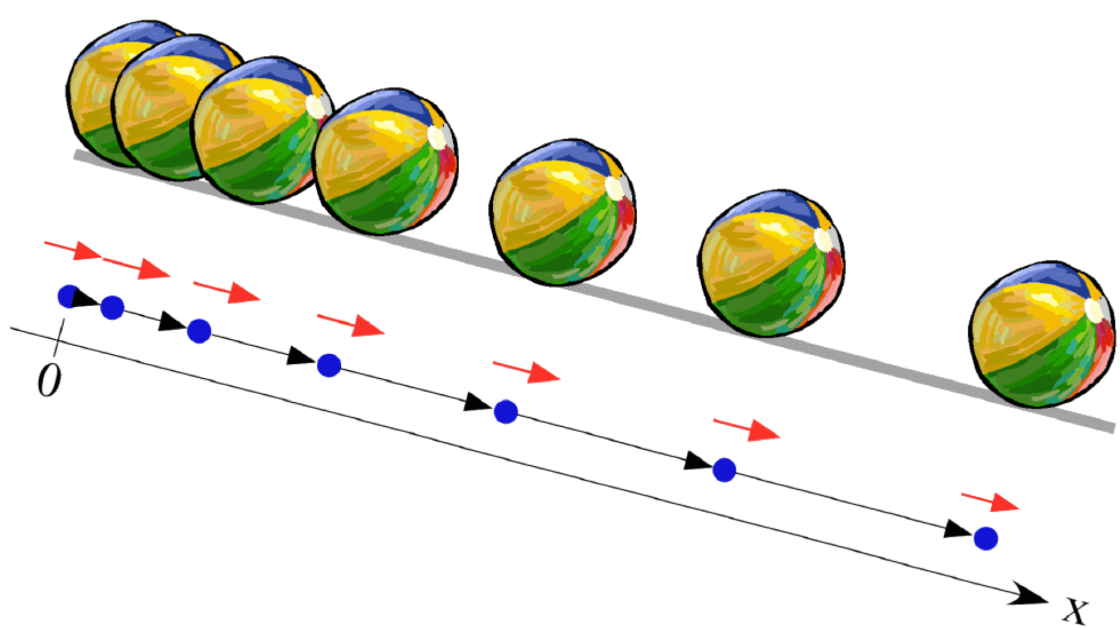
Example 3: An ostrich speeds up, trying to catch up with his group. (the speed of the ostrich is increasing). The acceleration is in the same direction as the ostrich's velocity.



Example 4: After participating in a 20 mile bike race, Tom approaches his home. His speed decreases as he prepares to stop. The direction of the acceleration is *opposite* to the direction of its velocity (Tom is slowing down).



Example 5: A ball rolls down a ramp, increasing its speed as it moves down the ramp. Note: motion diagrams are usually drawn parallel to the direction of motion.



Reading Page: Positive and Negative Velocities

Thus far most of our examples have dealt with positive values for velocity. However, velocity can have positive or negative values.

Remember that the slope of the x-t graph at any given time t_a gives you the value of the v-t graph at that same time t_a. In order to produce a v-t graph from an x-t graph, it is best to figure out the slope at several clock ticks on the x-t graph, and indicate them, one by one, on the v-t graph. (As you get more experienced, you might be able to do this by just moving your slope-o-meter - your pencil - like a tangent across the x-t graph).

Example 1. Object A starts at x=0 with zero velocity. It travels in the positive direction as shown in the x-t graph.

Let's take this step-by-step to produce a v-t graph from an x-t graph:

At any given time, the slope of the x-t graph gives the velocity at that time.

At t_1, the slope of the x-t graph at t_1, which gives the velocity at t_1 is zero. Therefore the velocity at t_1 is zero on the v-t graph.

At t_2, the slope of the x-t graph is a small positive value, and the velocity on the v-t graph is a small positive value.

At t_3, the slope of the x-t graph is larger than at t_2, and still positive; the velocity on the v-t graph at t_3 is larger than at t_2.

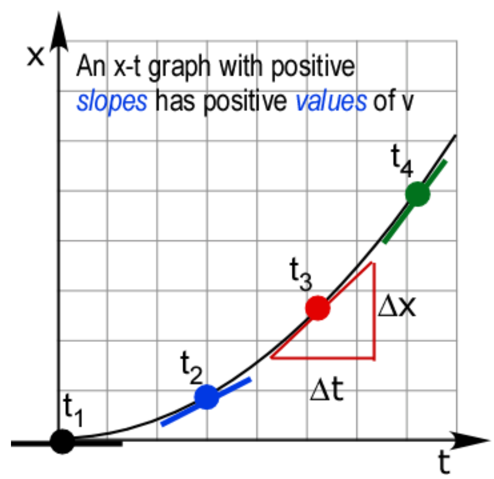
At t_4, the slope of the x-t graph has a large positive value, larger than at t_3,, indicating a large positive velocity, which is shown on the v-t graph.

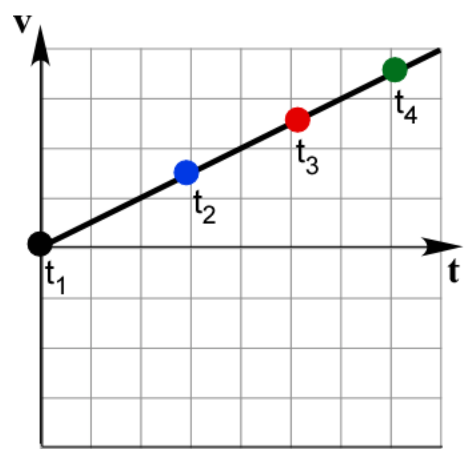
Since we have a smooth x-t graph, the values of the velocity change smoothly, giving us a v-t graph that is a straight line. It starts at v=0 and v increases to large positive values.

The slope of this straight v-t line gives the acceleration, which is also positive.

Since the values of velocity increase with time, the object speeds up. *Notice that both the velocity and the acceleration have positive values. Positive velocity means the object is traveling in the positive direction.*

Object A





Example 2. Object B moves in the positive direction, as shown in the x-t graph.

At t_1: the x-t graph has a large positive slope, and the value of velocity at t_1 on the v-t graph has a large positive value.

At t_2: The slope of the x-t graph decreases, but is still positive. The velocity at t_2 on the v-t graph is still a positive value, but less than that at t_1.

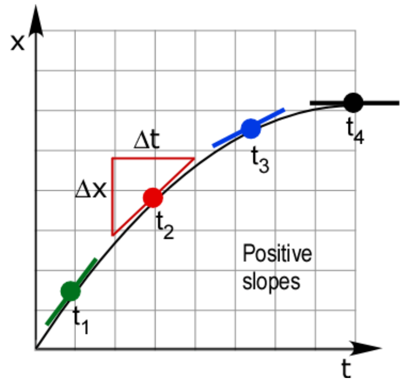
At t_3: the slope of the x-t graph decreases further. The velocity at t_3 on the v-t graph has a smaller value than at t_2.

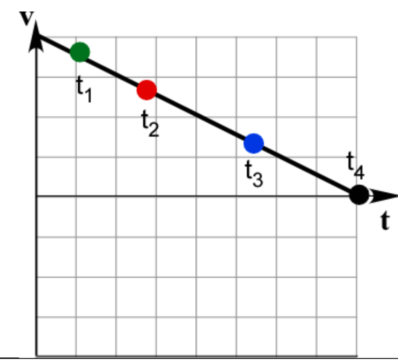
At t_4: the slope of the x-t graph is zero. The velocity at t_4 is zero.

The slope of this straight v-t line gives the acceleration, which is negative.

Since the values of velocity decrease with time, the object slows down. *Notice that the velocity has positive values but the acceleration has negative values. A positive velocity means the object is traveling in the positive direction.*

Object B.





Example 3: Object C moves toward the negative direction (toward the origin), as shown in the x-t graph.

At t_1: the slope of the x-t graph is zero. The velocity at t_1 is zero.

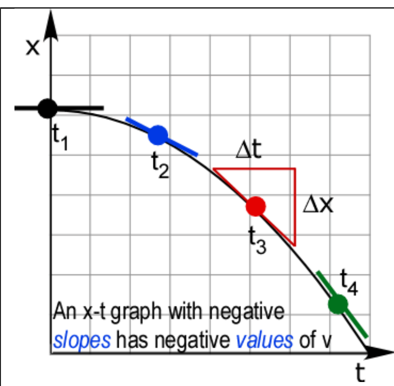
At t_2: The slope of the x-t graph is larger, but negative, The velocity at t_2 on the v-t graph is a negative value.

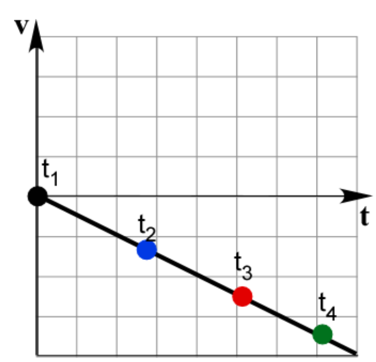
At t_3: the x-t graph has a larger negative slope, and the value of velocity at t_3 on the v-t graph has a larger negative value, larger than at t_2.

At t_4: the slope of the x-t graph increases further, but is still negative. The velocity at t_4 on the v-t graph has a larger negative value than at t_3.

The slope of this straight v-t line gives the acceleration, which is negative.

Since that the values (magnitudes) of the velocity get larger, this object speeds up. *Notice that the velocity and the acceleration both have negative values. A negative velocity means that the object is traveling in the negative direction.*





Example 4: Object D moves toward the negative direction (toward the origin), as shown in the x-t graph.

At t_1: the slope of the x-t graph is large and negative. The velocity at t_1 is large and negative.

At t_2: The slope of the x-t graph is smaller than at t_1, but still negative, The velocity at t_2 on the v-t graph is a smaller negative value than at t_1.

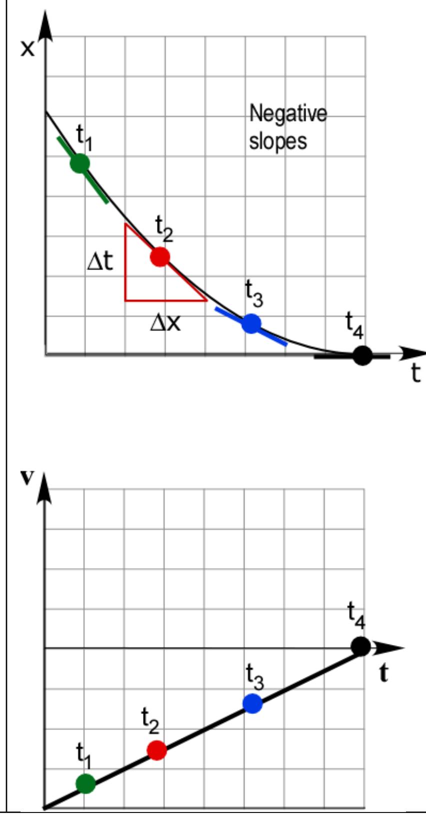
At t_3: the x-t graph has a even smaller negative slope than at t_2, and the value of velocity at t_3 on the v-t graph has a smaller negative value.

At t_4: the slope of the x-t graph decreases to zero. The velocity at t_4 on the v-t graph is zero.

The slope of this straight v-t line gives the acceleration, which is positive.

Since that the values (magnitudes) of the velocity get smaller, this object slows down. *Notice that the velocity has negative values but the acceleration has positive values. A negative velocity means that the object is traveling in the negative direction.*

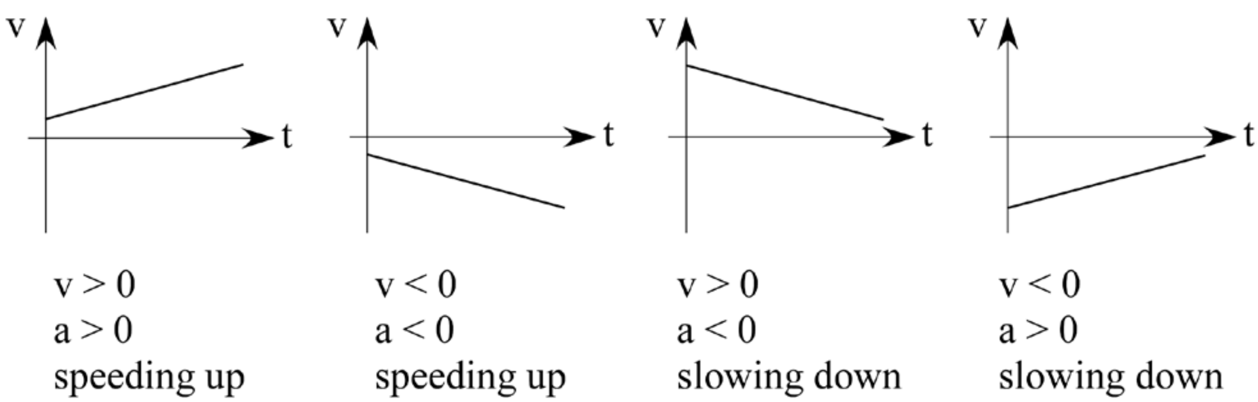
Object D:



These four examples illustrate the possible cases x-t graphs and the corresponding v-t graphs for uniformly accelerating objects.

Whether an object speeds up or slows down depends on the signs of *both* the velocity and the acceleration. If both are positive or both are negative, the object speeds up. If one is positive and one is negative, the object slows down.

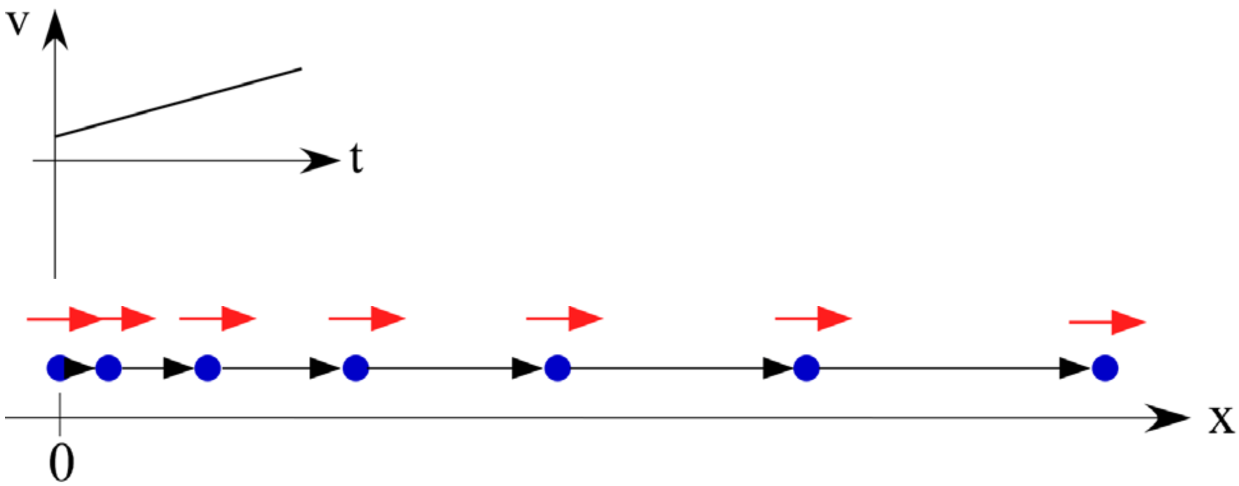
Positive acceleration does not necessarily mean speeding up, nor does negative acceleration mean slowing down. Instead, we need to correlate the signs of the velocity and the acceleration.



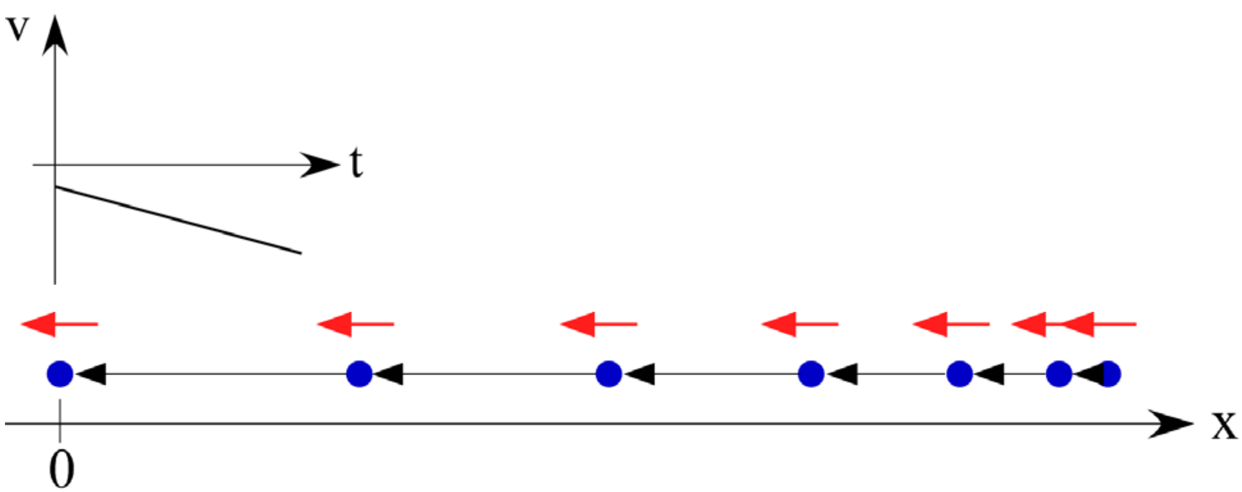
Reading Page: Motion Diagrams III

Correlating velocity versus time graphs to motion diagrams.

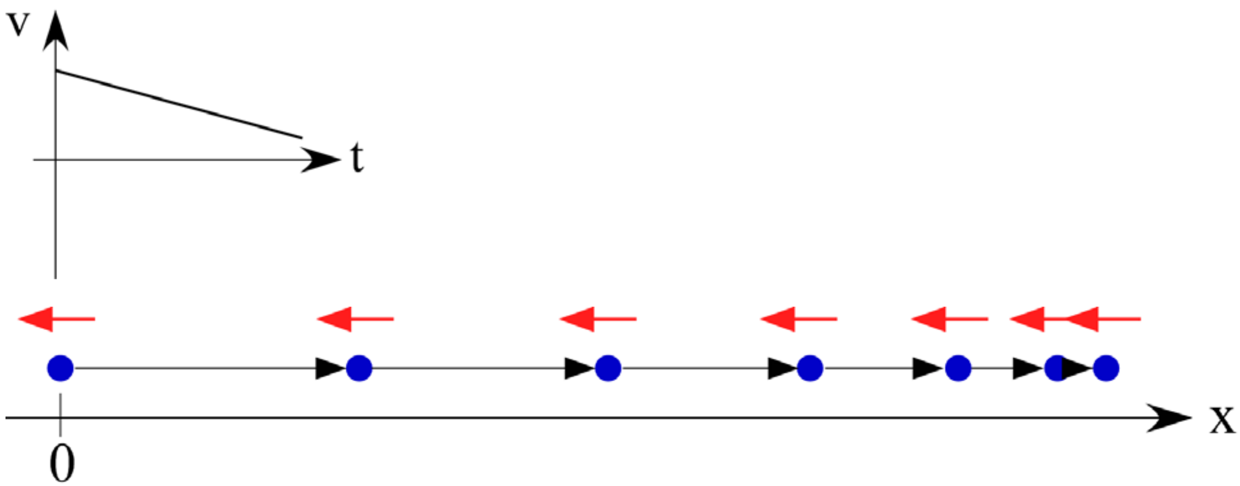
A. When an object moves toward the positive x direction, its velocity is positive. In the v *vs* t graph below, the velocity v starts at a small positive value and as the time increases, the value of the velocity increases. Its velocity increases as the object travels, which means that it is speeding up. The slope of the v *vs* t graph is positive, therefore the acceleration (which is the slope of the v *vs* t graph) is also positive. The slope is also constant over time, and therefore the acceleration is also constant over time.



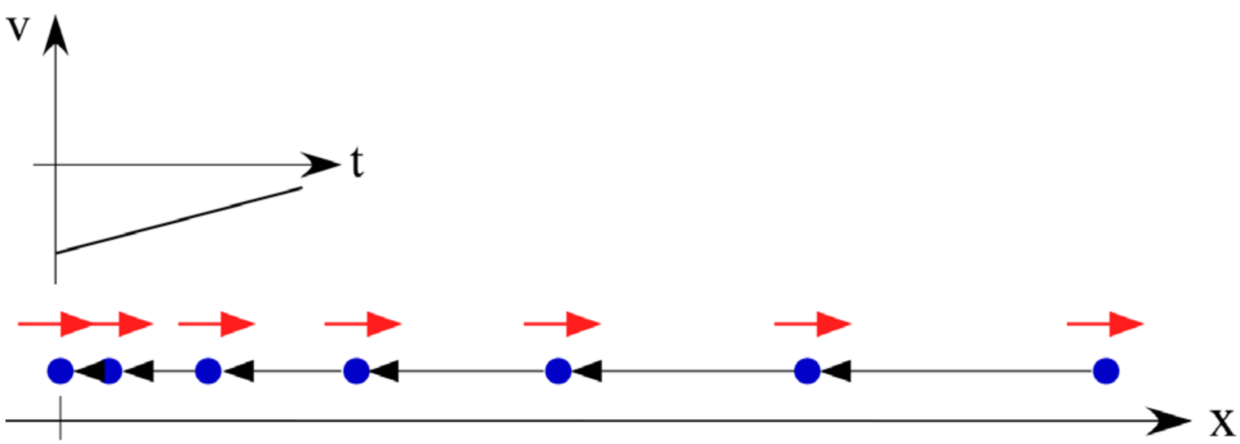
B. When an object moves toward the negative x direction, its velocity is negative. In the v *vs* t graph below, the object is moving in the negative direction (v<0) and the value of its velocity increases with time - so it is speeding up. A negative slope on a v *vs* t graph means negative acceleration. Speeding up means that both v and a have the same direction.



C. The object is moving in the positive direction (v>0) and slowing down: negative slope means negative acceleration. The velocity and acceleration have opposite directions.

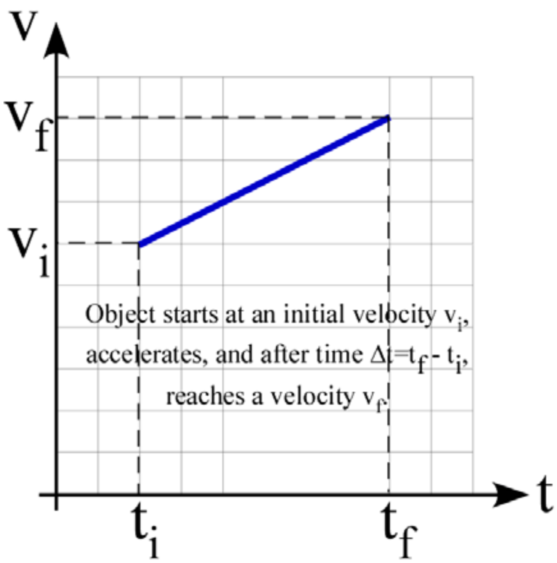


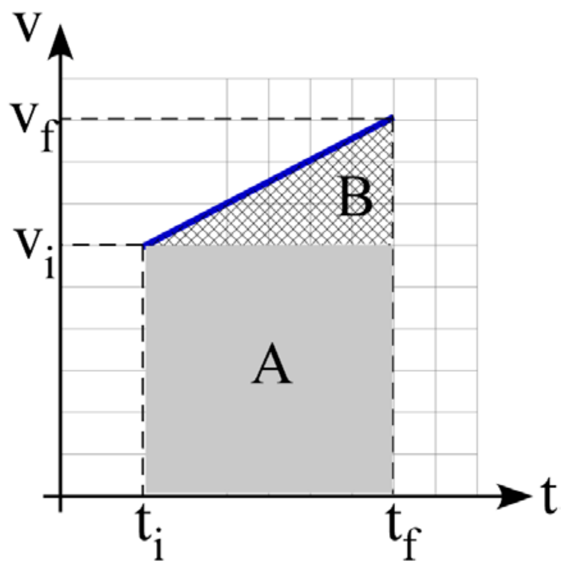
D. The object is moving to the left (v<0, velocity is negative) and slowing down: positive slope means positive acceleration. The velocity and acceleration have opposite directions.



Reading Page: Distance traveled while accelerating

If we have an object that has a starting (initial) velocity of v_i, which increases steadily to an ending (final) velocity of v_f, the v-t graph looks like the figure on the left below.





To find the displacement, all we need to do is to figure out the area under the line that describes the velocity. This area, as shown on the right, is equal to the area of the rectangle A plus area of the triangle B.

Displacement Δx = area of rectangle A + area of triangle B

$$\text{Area of triangle B} = \frac{1}{2}\text{base} \times \text{altitude}$$

$$= \frac{1}{2}\Delta t(v_f - v_i)$$

$$[\text{To simplify, since } v_f = v_i + a\Delta t, \text{ we can write } v_f - v_i = a\Delta t]$$

$$= \frac{1}{2}\Delta t(a\Delta t) = \frac{1}{2}a(\Delta t)^2$$

$$\text{Area of rectangle A} = v_i\Delta t$$

$$\text{Therefore, displacement } \Delta x = v_i\Delta t + \frac{1}{2}a(\Delta t)^2 \qquad \text{....Motion Equation \#2}$$

Reading Page: Examples - Using Motion Equations

When you are doing word problems that call for motion equations, it is best to define all your known quantities, identify the unknown quantity you are looking for, and then choose the appropriate motion equation. Here are the equations we know thus far:

Equation of motion#1: $v_f = v_i + a\Delta t$ (Eq. #1)

Equation of motion#2: $\Delta x = v_i \Delta t + \frac{1}{2} a(\Delta t)^2$ (Eq. #2)

Example 1. A go-cart starts traveling at the top of a hill at a speed of 7 meters/sec. It rolls down the hill for 5 seconds, after which it is traveling at 16 meters/sec. (a) Calculate its acceleration using the motion equations. (b) do the same problem graphically (c) Draw the motion diagram that describes the velocity of the go-cart.

(a) Solution using motion equations:

Choose units: speed in meters/sec, time in seconds.

List data and questions:

Initial speed v_i = 7 m/s

Final speed v_f = 16 m/s

Time taken Δt = 5 s

Acceleration a = ?

Identify formula and calculate:

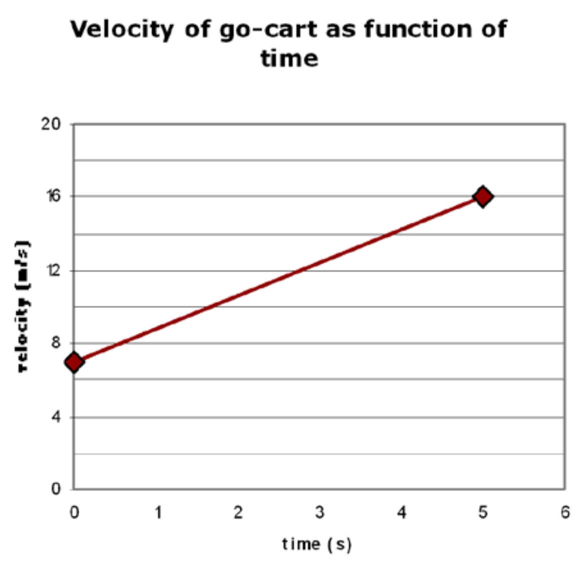
$$a = \frac{v_f - v_i}{\Delta t}$$

$$a = \frac{(16-7)(\frac{m}{s})}{5\text{s}} = \frac{9(\frac{m}{s})}{5\text{s}} = 1.8\frac{m}{s^2}$$

The go-cart has an acceleration of 1.8 m/s^2.

(b) *Graphical solution:*
The v-t graph is plotted on the right. Its slope gives the acceleration:

$$slope = acceleration,\ \text{a}=\frac{(16-7)\frac{m}{s}}{5s} = 1.8 m/s^2$$



Example 2. Generate a distance-time data table for the go-cart in example 1, assuming that the starting position is x=0. Use this table to draw a motion diagram of position x, velocity v, and acceleration a.

Time, Δt (s)	Calculate position: Use $\Delta x = v_i \Delta t + \frac{1}{2} a(\Delta t)^2$ with v_i =7 m/s; a=1.8 m/s^2	Calculated value of Δx, = x
0	=7(0) + (.5)(1.8)(0)2	0
1	=7(1) + (.5)(1.8)(1)2	7.9
2	=7(2) + (.5)(1.8)(2)2	17.6
3	=7(3) + (.5)(1.8)(3)2	29.1
4	=7(4) + (.5)(1.8)(4)2	42.4
5	=7(5) + (.5)(1.8)(5)2	57.5

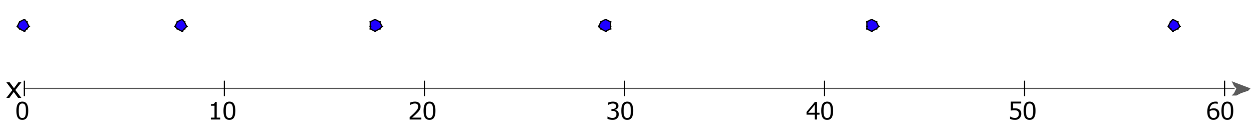
Solution:

Given values:
Initial velocity v_i = 7 m/s
Acceleration a = +1.8 m/s^2
Initial position x_i = 0
Position $x = x_i + \Delta x = 0 + \Delta x = \Delta x$

We can now generate a data table of x vs. Δt:

Motion diagram:
We draw here the motion diagram against a "ruler" so it is easy to see that the placement of the data points scales with the generated table.

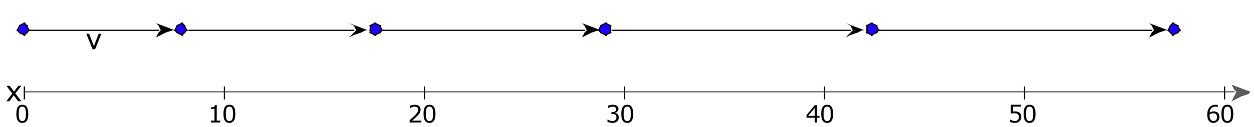
The first motion diagram we draw is the position, indicated by dots. Notice how the positions drawn match the values of x in the table above



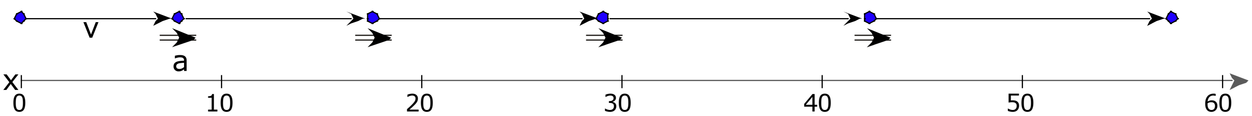
Next, we draw the velocities: we draw arrows between the dots. Geometrically, these arrows are proportional to the distance between the dots, e.g., $x_2 - x_1$ or $x_3 - x_2$. Since the average velocity in the time period between t_2 and t_1 is given by

$$v_{avg} = \frac{x_2 - x_1}{t_2 - t_1} \text{ during the time interval } t_2 - t_1,$$

the length of each arrow is proportional to the *average* velocity in each time intervals



Next, the acceleration is added to the motion diagram (double arrow). Since the acceleration is $(v_f - v_i)/\Delta t$, the length of this arrow is proportional to *difference* between successive velocity arrows, and is drawn centered on the clock tick. Notice that the difference between the successive velocity vectors is the same at all clock readings, showing that the value of the acceleration is the same throughout the motion. Since the velocity increases with time, the direction of the acceleration is the same as that of the velocity.



Example 3. A car is traveling down a street at 34 m/sec. The driver sees a slow tractor ahead and steps on her brake for 6 sec, which slows her down to 10 m/sec. (a) Calculate her acceleration using motion equations. (b) Calculate her acceleration graphically.

(a) Solution using motion equations:

Choose units: speed in meters/sec, time in seconds.

List data and questions:

Initial speed v_i = 34 m/s
Final speed v_f = 10 m/s
Time taken Δt=6 s
Acceleration a = ?

Calculate acceleration:

$$a = \frac{v_f - v_i}{t}$$

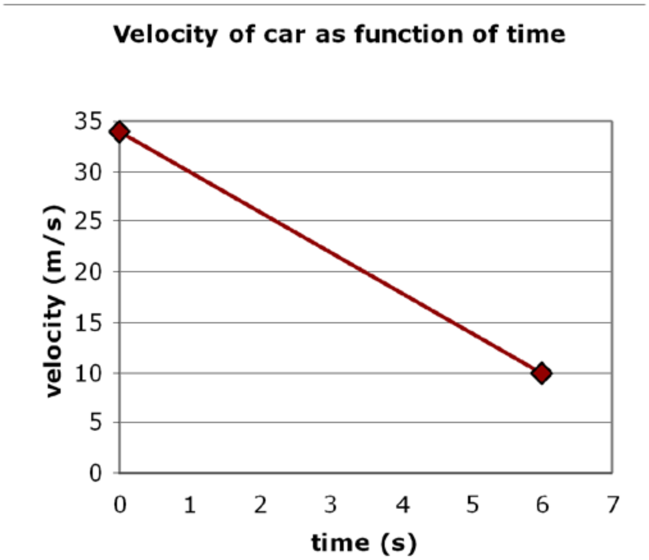
$$a = \frac{(10-34)(\frac{m}{s})}{6\,\text{s}} = \frac{-24(\frac{m}{s})}{6\,\text{s}} = -4.0\frac{m}{s^2}$$

Her acceleration is –4 m/s^2. Notice that her acceleration is *negative*, while her velocities are *positive*. On a coordinate axis, if her velocities point toward the right, her acceleration will point to the left. The *opposing directions* of velocity and acceleration indicate that she is *decelerating*.

(b) *Graphical Solution:*
The graph is shown on the right. The slope is:

$$\text{acceleration, } a = \frac{10-34}{6} = -4.0 m/s^2$$

Also notice the units for acceleration in both examples: the units are in meters/sec^2. The unit of acceleration is always in length/time^2.



Example 4. Generate a distance-time data table for the car in example 2, assuming that the starting position is x = 0 m. Use this table to draw a motion diagram of position x, velocity v, and acceleration a.

Time, Δt (s)	Calculate position: $\Delta x = v_i \Delta t + \frac{1}{2} a (\Delta t)^2$ with v_i =34 m/s; a=-4 m/s^2	Calculated value of Δx, = x (m)
0	=34(0) - (.5)(4)(0)^2	0
1	=34(1) - (.5)(4)(1)^2	32
2	=34(2) - (.5)(4)(2)^2	60
3	=34(3) - (.5)(4)(3)^2	84
4	=34(4) - (.5)(4)(4)^2	104
5	=34(5) - (.5)(4)(5)^2	120
6	=34(6) - (.5)(4)(6)^2	132

Check units:

Units of $v_i\Delta t$: $\frac{m}{\cancel{s}}\cancel{s} = m$

Units of $\frac{1}{2}a(\Delta t)^2 : \frac{m}{\cancel{s}^2}\cancel{s}^2 = m$

(The number $\frac{1}{2}$ has no units)

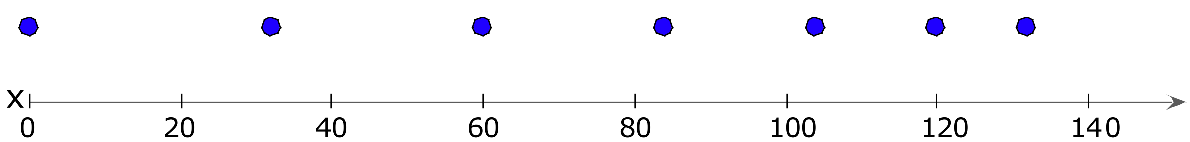
When we add two numbers, they must have the same units (or we can't add them).

Here $v_i\Delta t$ and $\frac{1}{2}a(\Delta t)^2$ both have units of meters, as they should.

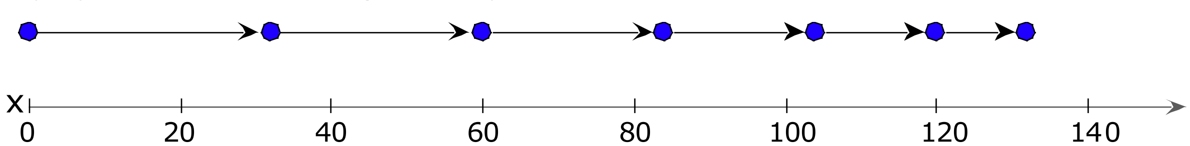
Solution:
Initial velocity v_i = 7 m/S
Acceleration a = +1.8 m/s^2
Initial position x_i = 0
Position $x = x_i + \Delta x = 0 + \Delta x = \Delta x$

We can now generate a data table of x vs. Δt:

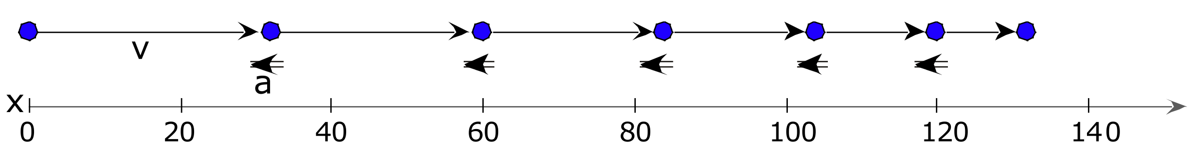
Motion diagram:
Again, we draw the motion diagram against a "ruler" so it is easy to see the placement of the data points. The first motion diagram we draw is the position, indicated by dots, with positions that match the values of x in the table above



Next, we draw the velocities as arrows between the dots. As in Example 2, the length of the arrow is proportional to the *average* velocity in that time interval.



Next, the acceleration is added to the motion diagram (double arrow). Again, notice that the difference between the successive velocity vectors is the same at all clock readings, showing that the value of the acceleration is the same throughout the motion. Since the velocity decreases with time, the direction of the acceleration is opposite to that of the velocity.



Example 5. Joni slides down the banisters at school (oops, that's not allowed, is it?) She pushes off with a speed of 10 cm/sec. She accelerates at a rate of 3 cm/sec^2 and it takes her 4 sec to reach the bottom of the banister. Calculate her speed when she gets to the bottom.

Solution:

Choose units: speed in cm/sec, time in seconds.

List data and questions:

Initial speed v_i = 10 cm/s

Time taken Δt = 4 s

Acceleration a = 3 cm/s^2

Final speed v_f = ?

Identify formula and calculate:

$$\begin{aligned} v_f &= v_i + a\Delta t \\ &= 10 + 3 \times 4 \\ &= 10 + 12 \\ &= 22 \text{ cm/s} \end{aligned}$$

Her speed at the bottom of the banister is 22 cm/s.

Example 6. Manya enters the highway at a speed of 36 m/sec. She steps on her accelerator for 12 sec, speeding up with an acceleration of 1.1 m/sec^2. How far did she travel during this time?

Choose units: When we have several different factors such as time, distance, speed and acceleration, it is usually convenient to choose a standard system of units such as the cm-gram-sec system or the meter-kg-sec system. When we do so, other factors we calculate come out in standard units, and we do not need to check on our units all the time. If we mixed units, such as meter-kg-hour, then we are using a non-standard system and we will have to keep track of our units, which is a lot more work.

For this example, it is convenient to use the meter-kg-sec system, where distance is in meters, time in sec, speed in m/sec, and acceleration in m/sec^2.

List data and questions:

Initial speed v_i=36 m/s

Time Δt = 12 s

Acceleration a = 1.1 m/s^2.

Distance Δx=?

Identify formula and calculate:

$$\Delta x = v_i \Delta t + \frac{1}{2} a(\Delta t)^2$$

$$\Delta x = (36 \times 12) + \frac{1}{2}(1.1) \times (12)^2$$

$$= 432 + 79.2$$

$$= 511.2m$$

Manya travels a distance of 511.2 m.

Example 7. Hillary pushes off on her skateboard at the top of a hill. She starts with a speed of 3 m/s, and gains speed steadily, accelerating at 0.8 m/s^2. She gets to the bottom of the hill 7 sec later. What is her speed at the bottom of the hill? How far did she travel?

Solution:

Choose units: meter-kg-sec system: distance in m, time in sec, speed in m/sec, acceleration in m/sec^2

List data and questions:

Initial speed v_i = 3 m/s

Acceleration a = 0.8 m/s^2

Time t = 7 s

Final speed v_f = ?

Distance Δx = ?

(Notice that two different questions were asked in this example.)

Identify formulae and calculate:

Final speed:

$$v_f = v_i + at$$

$$= 3 + 0.8 \times 7$$

$$= 8.6m / \mathrm{s}$$

Distance traveled:

$$\Delta x = v_i t + \frac{1}{2} at^2$$

$$\Delta x = (3 \times 7) + \frac{1}{2}(0.8) \times (7)^2$$

$$= 21 + 19.6$$

$$= 40.6m$$

Her final speed is 8.6 m/s and she has traveled 40.6 m.

The third equation of motion

So far we have used two equations of motion:

Equation of motion#1: $v_f = v_i + a\Delta t$ (Eq. #1)

Equation of motion#2: $\Delta x = v_i\Delta t + \frac{1}{2}a(\Delta t)^2$ (Eq. #2)

Eq. #1 connects initial and final speeds, acceleration and time. Eq. #2 connects distance, initial speed, acceleration and time. So far we do not have an equation that connects distance to final speed, or an equation that we can use if we do not know the time involved. We can get such an equation by combining Equations 1 and 2. If we use Eq. 1 to get an expression for the time Δt, and then substitute Δt into Eq. 2, we get (see the box for details):

Equation of motion #3: $v_f^2 - v_i^2 = 2a\Delta x$

Using these three equations of motion we can calculate pretty much anything we want to know about the motion of objects.

Example 8. In my newspaper I saw an advertisement for a car that could go from zero to 66m/sec in a distance of 400 m. Calculate its acceleration.

Choose units: meter-kg-sec.

List data and questions:

Initial speed v_i= 0

Final speed v_f= 66 m/s

Distance Δx = 400 m

Acceleration a = ?

Identify formula and calculate:

$$v_f^2 - v_i^2 = 2a\Delta x$$

We will have to manipulate this equation so that we get the acceleration a on one side of the equation:

Here's how we get Equation #3:

Start with Equation 1:

$$v_f = v_i + a\Delta t$$

Rearrange:

$$a\Delta t = v_f - v_i$$

Divide both sides by a:

$$\Delta t = \frac{v_f - v_i}{a},$$

which gives us an expression for Dt. Substitute this expression wherever Dt appears in Eq. #2:

$$\Delta x = v_i\left(\frac{v_f - v_i}{a}\right) + \frac{1}{2}a\left(\frac{v_f - v_i}{a}\right)^2$$

Simplify:

$$\Delta x = \left(\frac{v_i v_f - v_i^2}{a}\right) + \frac{1}{2}a\left(\frac{v_f^2 + v_i^2 - 2v_i v_f}{a^2}\right)$$

$$\Delta x = \left(\frac{v_i v_f - v_i^2}{a}\right) + \frac{1}{2}\left(\frac{v_f^2 + v_i^2 - 2v_i v_f}{a}\right)$$

$$a\Delta x = v_i v_f - v_i^2 + \frac{v_f^2}{2} + \frac{v_i^2}{2} - v_i v_f$$

$$a\Delta x = \frac{v_f^2}{2} - \frac{v_i^2}{2}$$

$$2a\Delta x = v_f^2 - v_i^2$$

$$2a\Delta x = v_f^2 - v_i^2$$

$$a = \frac{v_f^2 - v_i^2}{2\Delta x} = \frac{(66)^2 - 0^2}{2 \times 400}$$

$$= 5.44 m/\mathrm{s}^2$$

The car has an acceleration of 5.44 m/s^2.

Summary: Here are the three equations of motion. You will have to identify which of these three is appropriate for each situation:

Equations of Motion:	***Unit system***		
	Here	*cm-g-s:*	*m-kg-s*
Equation #1: $v_f = v_i + a\Delta t$	Δx = distance	cm	m
	Δt = time	sec	sec
Equation #2: $\Delta x = v_i\Delta t + \frac{1}{2}a(\Delta t)^2$	v_i = initial speed	cm/s	m/s
Equation #3: $v_f^2 - v_i^2 = 2a\Delta x$	v_f = final speed	cm/s	m/s
	a= acceleration	cm/s^2	m/s^2